On the Quaternary Linear Homogeneous Groups Modulo p of Order a Multiple of p.

By Leonard Eugene Dickson.*

1. Consider the group G of all quaternary linear homogeneous transformations modulo p of determinant unity. The order of G is

$$\tau p^6, \qquad \tau \equiv (p^4-1)(p^3-1)(p^2-1). \tag{1}$$

Every subgroup of order a power of p is conjugate within G with a subgroup of the group G_{p^6} of the operators

$$[\alpha_{ij}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{pmatrix} \tag{2}$$

The subgroups of this G_{p^6} are given in §3. Some of these were determined in in an earlier paper†, the others by similar methods.

Let H denote any subgroup of order p^hN of G, N being prime to p, $h>0$. Applying a suitable transformation within G, we may assume that H contains a subgroup G_{p^h} of G_{p^6}. If H contains G_{p^h} self-conjugately, H lies in the corresponding group determined in §4, and the explicit determination of H is essentially a problem on binary or ternary groups. Suppose next that G_{p^h} is not self-conjugate in H and let p^m be the maximal order of a subgroup common to G_{p^h} and any of its conjugates under H; let G_{p^m} be such a subgroup. By a theorem‡ due to Burnside and Frobenius, H must contain an operator S, of period prime to p, commutative with G_{p^m} but not with G_{p^h}.

By §2, τ contains no factor of the form $1+p^4x$, $x>0$. Hence the number of conjugates to G_{p^h} in H is not $\equiv 1 \pmod{p^4}$, so that|| $m \geqq h-3$. Further, if

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† *American Journal of Mathematics*, vol. 27 (1905), pp. 280-302.

‡ References in Burnside's *Theory of Groups*, p. 97.

|| Compare Burnside's *Theory of Groups*, p. 94, Cor. II.

$p > 2$ and if the number of conjugates to G_{p^h} is not $(p^4-1)(p^3-1)$, we may (§2) set $m \overline{\gtrless} h-2$. To these results we owe the practicability of the method employed in the problem. The general theorem of §7 leads to a considerable reduction in the number of cases to be treated.

For $p=2$, G is simply isomorphic with the alternating group on 8 letters.* When exceptional, the case $p=2$ is excluded.

2. LEMMA. *The only factor of the form* $1+p^3x$, $x>0$, *of* τ *is* $(p^4-1)(p^3-1)$, *an additional factor* 9 *occurring if* $p=2$.

Let $\tau = (1+p^3x)q$. Then $p^2-1 \equiv q \pmod{p^3}$. Let $q = mp^3+p^2-1$, $m \overline{\gtrless} 0$. For $m=0$, we reach $(p^4-1)(p^3-1)$. Let henceforth $m \overline{\gtrless} 1$. Then

$$q > p^3,\ 1+p^3x = \tau/q < (p^9-p^7)/p^3,\ x < p^3-p.$$

Expanding $\tau-(1+p^3x)q$ and dividing by p^3, we get

$$p^6-p^4-p^3-p^2+p+1-mxp^3-p^2x+x-m=0.$$

Hence there is an integer t such that

$$m = x+p+1+tp^2. \tag{3}$$

Then $p^4-p^2-p-1-mxp-x-t=0$. Hence for some integer s,

$$x+t = sp-1,\ p^3-p-1-mx-s=0. \tag{4}$$

If $t>0$, then $s>0$, $m>tp^2$, $q>tp^5$, $1+p^3x<p^4/t$, $x<p/t$. Hence $x<p$, $t<p$. Then by $(4)_1$, $sp-1 \overline{\lessgtr} 2p-2$, $s=1$, $p-1<t+p/t$. If $p>3$, $(2t-p)^2 \overline{\gtrless} p^2-4p > (p-3)^2$, requiring $t=1$. If $p=3$, then $x=t=1$. In every case, $s=1, t=1, x=p-2, m=p^2+2p-1$, so that $(4)_2$ gives $p=1$.

If $t=0$, $x=sp-1$, $s>0$, $m=sp+p$. Then $(4)_2$ becomes

$$p^3-1-s^2p^2-sp^2+sp-s=0,$$

whence $p^3-s^2p^2>0$, $s=-1+wp \overline{\gtrless} p-1$. Thus $p=2$, $s=1$, $x=1$.

If $t=-r$, where $r>0$, (3) gives $x \overline{\gtrless} rp^2-p$. Then by $(4)_1$, $s>0$. But $x<p^3-p$. Hence $r<p$. Set $x=rp^2-p+z$, $z \overline{\gtrless} 0$. Then $m=z+1$, by (3), and $sp=rp^2-p+z+1-r$ by (4). Hence

$$z+1-r = lp,\ s = rp-1+l.$$

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* References and proof in *Linear Groups*, §268, p. 290.

By the first, $l \overline{>} 0$ since $r < p$. For $l = 0$, (4) becomes

$$p^3 - p = r^2p^2 + r^2 - r,$$

whence $r \equiv 0$ or $1 \pmod p$, $r = 1$, which is impossible. Let next $l \overline{>} 1$. Now $(4)_1$ may be written

$$p^3 - p - 1 = (r + lp)(rp^2 - p + lp + r - 1) + rp - 1 + l.$$

But the second member $> p\,(rp^2)$, the first member $< p^3$.

3. The $p^2 + p + 1$ subgroups of order p^5 of G_{p^6} are given by a linear homogeneous relation between α_{21}, α_{32}, α_{43}. For each G_{p^5} given in the first column of the following table, the subgroups of order p^4 are obtained by annexing a linear homogeneous relation between the elements in the second column:

$\{\alpha_{21} = 0\}$	α_{31}, α_{32}, α_{43}
$\{\alpha_{32} = 0\}$	α_{21}, α_{43}, α_{42}, α_{31}
$\{\alpha_{43} = 0\}$	α_{21}, α_{32}, α_{42}
$\{c_2\alpha_{21} + c_3\alpha_{32} + c_4\alpha_{43} = 0\}$	α_{21}, α_{32}, α_{43}, ϕ

where for the last G_{p^5}, at least two c_i are $\neq 0$, $p > 2$, and

$$\phi = c_2\alpha_{31} - c_4\alpha_{42} - \tfrac{1}{2}c_2\alpha_{32}\alpha_{21} + \tfrac{1}{2}c_4\alpha_{43}\alpha_{32}.$$

For $p > 2$, the $p^4 + 2p^3 + 2p^2 + p + 1$ distinct G_{p^4} are given in the first column of the following table; the subgroups of order p^3 of each G_{p^4} are obtained by annexing a linear relation between the elements in the second column:

$\{\alpha_{21} = 0,\ \alpha_{43} = 0\}$	α_{31}, α_{32}, α_{41}, α_{42}
$\{\alpha_{43} = 0,\ \alpha_{32} = s\alpha_{21}\}$	α_{21}, α_{42}, $a = \alpha_{31} - \frac{1}{2}s\alpha_{21}^2$
$\{\alpha_{21} = r\alpha_{43},\ \alpha_{32} = s\alpha_{43}\}$	α_{43}, $b = \alpha_{42} - \frac{1}{2}s\alpha_{43}^2$, $c = \alpha_{31} - \frac{1}{2}rs\alpha_{43}^2$
$\{\alpha_{21} = 0,\ \alpha_{31} = r\alpha_{32} + s\alpha_{43}\}$	α_{32}, α_{43}, $d = \alpha_{41} - r\alpha_{42} - \frac{1}{2}s\alpha_{43}^2$
$\{\alpha_{43} = 0,\ \alpha_{42} = r\alpha_{32} + s\alpha_{21}\}$	α_{21}, α_{32}, $e = \alpha_{41} - r\alpha_{31} - \frac{1}{2}s\alpha_{21}^2$
$\{\alpha_{32} = 0,\ \alpha_{31} = r\alpha_{21} + s\alpha_{43}\}$	α_{21}, α_{43}, α_{42}
$\{\alpha_{32} = 0,\ \alpha_{42} = r\alpha_{21} + s\alpha_{43} + t\alpha_{31}\}$	α_{21}, α_{43}, α_{31}
$\left\{\sum_{i=2}^{4} c_i\alpha_{i\,i-1} = 0,\ \phi + \sum_{i=2}^{4} k_i\alpha_{i\,i-1} = 0\right\}$	α_{21}, α_{32}, α_{43}

For $p > 2$, the $4p^4 + 3p^3 + 2p^2 + p + 1$ distinct G_{p^3} are given in the first column of the following table, the subgroups of order p^2 of each G_{p^3} are obtained

by annexing a linear relation between the elements in the second column:

$\{\alpha_{21} = \alpha_{43} = 0,\ \text{lin.}(\alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42}) = 0\}$	$\alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42}$
$\{\alpha_{43} = 0,\ \alpha_{32} = s\alpha_{21},\ a = r\alpha_{21} + t\alpha_{42}\}$	α_{21}, α_{42}
$\{\alpha_{43} = 0,\ \alpha_{32} = s\alpha_{21},\ \alpha_{42} = r\alpha_{21}\}$	$\alpha_{21}, a, f = \alpha_{41} - \frac{1}{2} r\alpha_{21}^2$
$\left\{\begin{matrix}\alpha_{21} = r\alpha_{43},\ \alpha_{32} = s\alpha_{43}, \\ b = tc + u\alpha_{43}\end{matrix}\right\}$	α_{43}, c, if $rt \neq 1$, or $rt = 1, p = 3, s \neq 0$ α_{43}, c, g, if $rt = 1$, with $s = 0$ when $p = 3$
$\left\{\begin{matrix}\alpha_{21} = r\alpha_{43},\ \alpha_{32} = s\alpha_{43}, \\ c = t\alpha_{43}\end{matrix}\right\}$	α_{43}, b, if $r \neq 0$ $\alpha_{43}, b, h = \alpha_{41} - \frac{1}{2} t\alpha_{43}^2$, if $r = 0$.
$\left\{\begin{matrix}\alpha_{21} = 0,\ \alpha_{31} = r\alpha_{32} + s\alpha_{43}, \\ d = t\alpha_{32} + u\alpha_{43}\end{matrix}\right\}$	α_{32}, α_{43}
$\left\{\begin{matrix}\alpha_{43} = 0,\ \alpha_{42} = r\alpha_{32} + s\alpha_{21}, \\ e = t\alpha_{21} + u\alpha_{32}\end{matrix}\right\}$	α_{21}, α_{32}
$\left\{\begin{matrix}\alpha_{32} = 0,\ \alpha_{31} = r\alpha_{21} + s\alpha_{43}, \\ \alpha_{42} = t\alpha_{21} + u\alpha_{43}\end{matrix}\right\}$	α_{21}, α_{43}, if $r + u \neq 0$ $\alpha_{21}, \alpha_{43}, j$, if $r + u = 0$

where the further abbreviations have been used

$$g = \alpha_{41} - \alpha_{31}\alpha_{43} + \tfrac{1}{3} rs\alpha_{43}^3 - \tfrac{1}{2} ru\alpha_{43}^2,\quad j = \alpha_{41} - \tfrac{1}{2} t\alpha_{21}^2 - \tfrac{1}{2} s\alpha_{43}^2.$$

For $p > 2$, the distinct G_{p^2} are given in the first column of the next table;

$$2p^5 + 3p^4 + 2p^3 + 2p^2 + p + 1\ (p > 3),\ p^5 + 5p^4 + p^3 + 2p^2 + p + 1\ (p = 3)$$

is the number of G_{p^2}; the subgroups C_p of each G_{p^2} are obtained by annexing a linear relation between the elements of the second column:

$\{\alpha_{21} = \alpha_{43} = 0,\ \text{two lin.}\ (\alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42})\}$	$\alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42}$
$\{\alpha_{43} = 0,\ \alpha_{32} = s\alpha_{21},\ a = \rho\alpha_{21},\ \alpha_{42} = r\alpha_{21}\}$	α_{21}, f
$\{\alpha_{43} = 0,\ \alpha_{32} = s\alpha_{21},\ \alpha_{42} = r\alpha_{21},\ f = ta + u\alpha_{21}\}$	α_{21}, a
$\left\{\begin{matrix}\alpha_{21} = r\alpha_{43},\ \alpha_{32} = s\alpha_{43},\ b = \rho\alpha_{43}, \\ c = w\alpha_{43}\end{matrix}\right\}$	$\alpha_{43} = 0$, if $p = 3,\ rs \neq 0$ α_{43}, k, if $p > 3$, or $p = 3,\ rs = 0$
$\left\{\begin{matrix}\alpha_{21} = r\alpha_{43},\ b = r^{-1}c + u\alpha_{43}, \\ \alpha_{32} = s\alpha_{43},\ g = wc + z\alpha_{43},\end{matrix}\ s = 0 \text{ if } p = 3\right\}$	α_{43}, c
$\{\alpha_{21} = 0,\ \alpha_{32} = s\alpha_{43},\ \alpha_{31} = t\alpha_{43},\ h = wb + z\alpha_{43}\}$	α_{43}, b
$\left\{\begin{matrix}\alpha_{32} = 0,\ \alpha_{31} = r\alpha_{21} + s\alpha_{43}, \\ \alpha_{42} = t\alpha_{21} - r\alpha_{43},\ j = w\alpha_{21} + z\alpha_{43}\end{matrix}\right\}$	α_{21}, α_{43}

where $k = \alpha_{41} - \frac{1}{6} rs\alpha_{43}^3 - \frac{1}{2}(w + \rho r)\alpha_{43}^2$. The only distinct C_p in this table are the $p^3 + p^2 + p + 1$ groups $\{\alpha_{21} = \alpha_{43} = 0$, three lin. $(\alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42})\}$, the p^4 groups

$$\{\alpha_{43} = 0,\ \alpha_{32} = s\alpha_{21},\ a = \rho\alpha_{21},\ \alpha_{42} = r\alpha_{21},\ f = w\alpha_{21}\}$$

and the p^5 $(p > 3)$ or $5p^3$ $(p = 3)$ groups, with $rs = 0$ if $p = 3$,

$$\{\alpha_{21} = r\alpha_{43},\ \alpha_{32} = s\alpha_{43},\ b = \rho\alpha_{43},\ c = w\alpha_{43},\ k = z\alpha_{43}\}.$$

Hence there are $p^5 + p^4 + p^3 + p^2 + p + 1$ $(p > 3)$, $p^5 + p^2 + p + 1 = 256$ $(p = 3)$ distinct C_p in G_{p^6}, as is also evident directly.

4. In the first column of the following table appears a representative* of each set of conjugate subgroups of order a power of p of G, for $p > 2$. All quaternary transformations (δ_{ij}) commutative with this representative are given in the second column. The conditions for determinant unity are omitted for several reasons; they may be supplied by inspection. Here $\mu = 1$ or a particular not-square ν; $l = 0,\ 1,\ \nu$; ν_1 is any not-square.

G_{p^6}	$\delta_{ij} = 0\ (j > i)$
$\{\alpha_{ss-1} = 0\}$, $s = 2,\ 3$, or 4	$\delta_{ij} = 0\ (j > i)$ except δ_{s-1s}
$\{\alpha_{21} = \alpha_{32} + \mu\alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{22}^2 = \delta_{11}\delta_{33}$, $\delta_{33}^2 = \delta_{22}\delta_{44}$
$\{\alpha_{21} = \mu\alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{11}\delta_{44} = \delta_{22}\delta_{33}$
$\{\alpha_{21} = \alpha_{32}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{22}^2 = \delta_{11}\delta_{33}$
$\{\alpha_{32} = \alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{33}^2 = \delta_{22}\delta_{44}$
$\{\alpha_{21} = \alpha_{32} = 0\}$, $\{\alpha_{32} = \alpha_{43} = 0\}$	$\delta_{ij} = 0\ (j > i)$
$\{\alpha_{21} = \alpha_{43} = 0\}$	$\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0$
$\{\alpha_{21} = 0,\ \alpha_{32} = \alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{33}^2 = \delta_{22}\delta_{44}$
$\{\alpha_{32} = 0,\ \alpha_{21} = \mu\alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{11}\delta_{44} = \delta_{22}\delta_{33}$
$\{\alpha_{43} = 0,\ \alpha_{21} = \alpha_{32}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{22}^2 = \delta_{11}\delta_{33}$
$\{\alpha_{21} = \mu\alpha_{43},\ \alpha_{32} = \alpha_{43}\}$	" $\delta_{22}^2 = \delta_{11}\delta_{33}$, $\delta_{33}^2 = \delta_{22}\delta_{44}$
$\{\alpha_{32} = 0,\ \alpha_{42} = \alpha_{21}\}$	" $\delta_{32} = 0$, $\delta_{22}^2 = \delta_{11}\delta_{44}$
$\{\alpha_{32} = \alpha_{43},\ \alpha_{42} = \frac{1}{2}\alpha_{43}^2 + \alpha_{21}\}$	" $\delta_{32}\delta_{44} = \delta_{43}\delta_{33}$, $\delta_{33}^2 = \delta_{22}\delta_{44}$, $\delta_{22}^2 = \delta_{11}\delta_{44}$
$\{\alpha_{32} = \alpha_{43},\ \alpha_{42} = \frac{1}{2}\alpha_{43}^2\}$	" " "
$\{\alpha_{21} = \alpha_{32},\ \alpha_{31} = \frac{1}{2}\alpha_{32}^2 + \alpha_{43}\}$	" $\delta_{21}\delta_{33} = \delta_{32}\delta_{22}$, $\delta_{22}^2 = \delta_{11}\delta_{33}$, $\delta_{33}^2 = \delta_{11}\delta_{44}$
$\{\alpha_{21} = \alpha_{32},\ \alpha_{31} = \frac{1}{2}\alpha_{32}^2\}$	" " "
$\left\{\begin{matrix}\alpha_{21} = \mu\alpha_{43} \\ \alpha_{31} + \mu\alpha_{42} = \mu\alpha_{32}\alpha_{43} + \kappa\alpha_{43}\end{matrix}\right\}$	" $\delta_{21}\delta_{33} = \mu\delta_{43}\delta_{11}$, $\delta_{22}\delta_{33} = \delta_{11}\delta_{44}$ also $\delta_{33}^2 = \delta_{11}\delta_{44}$ if $\kappa \neq 0$
$\left\{\begin{matrix}\alpha_{21} = \alpha_{32} + \mu\alpha_{43} \\ \alpha_{31} + \mu\alpha_{42} = \frac{1}{2}\alpha_{32}^2 + \mu\alpha_{32}\alpha_{43} + \kappa\alpha_{43}\end{matrix}\right\}$	" $\delta_{21}\delta_{33} = \delta_{32}\delta_{22} + \mu\delta_{43}\delta_{11}$, $\delta_{22}\delta_{33} = \delta_{11}\delta_{44}$, $\delta_{22}^2 = \delta_{11}\delta_{33}$; also $\delta_{33} = \delta_{22}$ if $\kappa \neq 0$

* I have aimed to list but one representative, but I have relied mostly upon inspection in this direction. However I have taken every precaution and check to make sure that every set of conjugates is represented in the list.

$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=0\}$	$\delta_{ij}=0\ (j>i)$
$\{\alpha_{21}=\alpha_{32}=\alpha_{31}=0\}$	$\delta_{14}=\delta_{24}=\delta_{34}=0$
$\{\alpha_{21}=\alpha_{32}=\alpha_{42}=0\}$	$\delta_{ij}=0\ (j>i),\ \delta_{32}=0$
$\{\alpha_{32}=\alpha_{43}=\alpha_{42}=0\}$	$\delta_{12}=\delta_{13}=\delta_{14}=0$
$\{\alpha_{21}=\alpha_{32}=0,\ \alpha_{31}=\alpha_{43}\}$	$\delta_{ij}=0\ (j>i),\ \delta_{33}^2=\delta_{11}\delta_{44}$
$\{\alpha_{21}=\alpha_{32}=0,\ \alpha_{31}=\mu\alpha_{42}\}$	“ $\delta_{32}=0,\ \delta_{22}\delta_{33}=\delta_{11}\delta_{44}$
$\{\alpha_{32}=\alpha_{43}=0,\ \alpha_{31}=\mu\alpha_{42}\}$	“ “ “
$\{\alpha_{32}=\alpha_{43}=0,\ \alpha_{42}=\alpha_{21}\}$	“ $\delta_{22}^2=\delta_{11}\delta_{44}$
$\{\alpha_{21}=\alpha_{43}=0,\ \alpha_{41}=\mu\alpha_{32}\}$	$\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=0$ and (5)
$\left\{\begin{array}{l}\alpha_{21}=0,\ \alpha_{32}=\alpha_{43}\\ \alpha_{42}-\frac{1}{2}\alpha_{43}^2=l\alpha_{31}\end{array}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{32}\delta_{44}\delta_{33}=\delta_{43}\delta_{33}^2+l\delta_{21}\delta_{44}^2.$ $\delta_{33}^2=\delta_{22}\delta_{44}$; also $\delta_{11}\delta_{44}=\delta_{22}\delta_{33}$ if $l\neq 0$
$\left\{\begin{array}{l}\alpha_{32}=0,\ \alpha_{21}=\mu\alpha_{43}\\ \alpha_{31}=-\mu\alpha_{42}+l\alpha_{43}\end{array}\right\}$	$\delta_{12}=\delta_{13}=\delta_{14}=\delta_{24}=\delta_{34}=0,\ \delta_{22}\delta_{33}-\delta_{23}\delta_{32}=\delta_{11}\delta_{44}\ (l=0)$ $\delta_{ij}=0\ (j>i),\ \delta_{22}\delta_{33}=\delta_{11}\delta_{44}=\delta_{33}^2\ (l\neq 0)$
$\left\{\begin{array}{l}\alpha_{32}=0,\ \alpha_{21}=\mu\alpha_{43}\\ \alpha_{31}=m\alpha_{42},\ m\neq 0,\ m\neq-\mu\end{array}\right\}$	“ , $\delta_{32}=0,\ \delta_{22}\delta_{33}=\delta_{11}\delta_{44}\ (m\neq\mu)$ Preceding (δ_{ij}) and $\delta_{23}\delta_{32}=\delta_{11}\delta_{44}$, $\delta_{12}=\delta_{13}=\delta_{14}=\delta_{22}=\delta_{24}=\delta_{33}=\delta_{34}=0\ (m=\mu)$
$\left\{\begin{array}{l}\alpha_{43}=0,\ \alpha_{21}=\alpha_{32}\\ \alpha_{31}-\frac{1}{2}\alpha_{32}^2=l\alpha_{42}\end{array}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{22}\delta_{32}=\delta_{33}\delta_{21}+l\delta_{11}\delta_{43},$ $\delta_{22}^2=\delta_{11}\delta_{33}$; also $\delta_{22}\delta_{33}=\delta_{11}\delta_{44}$ if $l\neq 0$
$\{\alpha_{21}=\mu\alpha_{43},\ \alpha_{32}=\alpha_{43},\ \alpha_{42}=\frac{1}{2}\alpha_{43}^2\}$	$\delta_{ij}=0\ (j>i),\ \delta_{44}\delta_{11}=\delta_{22}\delta_{33},\ \delta_{33}^2=\delta_{22}\delta_{44},\ \delta_{33}\delta_{43}=\delta_{44}\delta_{32}$
$\left\{\begin{array}{l}\alpha_{21}=\mu\alpha_{43},\ \alpha_{32}=\alpha_{43}\\ \alpha_{31}-\frac{1}{2}\mu\alpha_{43}^2=m\ (\alpha_{42}-\frac{1}{2}\alpha_{43}^2)\end{array}\right\}$	“ , $\delta_{33}^2=\delta_{22}\delta_{44},\ \delta_{22}^2=\delta_{11}\delta_{33},$ $\delta_{21}\delta_{44}=(\mu+m)\ \delta_{32}\delta_{33}-m\delta_{43}\delta_{22}$
$\left\{\begin{array}{r}\alpha_{32}=0,\alpha_{42}=\alpha_{21},\ \alpha_{31}=q\alpha_{21}+m\alpha_{43}\\ q^2-4m\text{ a not-square}\end{array}\right\}$	$\delta_{12}=\delta_{13}=\delta_{14}=\delta_{24}=\delta_{34}=0$ and (6)

$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=\alpha_{41}=0\}$	$\delta_{i}=0\ (j>i),\ \delta_{21}=\delta_{43}=0$; prod. by (12)(34)
$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=\alpha_{42}=0\}$	$\delta_{12}=\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=0$
$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=\alpha_{31}=0\}$	$\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=\delta_{34}=0$
$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=0,\ \alpha_{42}=\mu\alpha_{31}\}$	$\delta_{ij}=0\ (j>i),\ \delta_{11}\delta_{44}=\delta_{22}\delta_{33}$
$\{\alpha_{21}=\alpha_{32}=\alpha_{42}=0,\ \alpha_{31}=\alpha_{43}\}$	“ , $\delta_{32}=0,\ \delta_{33}^2=\delta_{11}\delta_{44}$
$\{\alpha_{21}=\alpha_{32}=0,\ \alpha_{31}=\alpha_{43},\ \alpha_{41}=\frac{1}{2}\alpha_{43}^2\}$	“ , $\delta_{21}=0,\ \delta_{33}^2=\delta_{11}\delta_{44},\ \delta_{11}\delta_{43}=\delta_{33}\delta_{31}$
$\{\alpha_{32}=\alpha_{43}=0,\ \alpha_{42}=\alpha_{21},\ \alpha_{41}=\frac{1}{2}\alpha_{21}^2\}$	“ , $\delta_{43}=0,\ \delta_{22}^2=\delta_{11}\delta_{44},\ \delta_{11}\delta_{42}=\delta_{22}\delta_{21}$
$\{\alpha_{21}=\alpha_{43}=0,\ \alpha_{41}=\alpha_{32},\ \alpha_{42}=\nu\alpha_{31}\}$	$\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=0$*, $\delta_{11}=\pm\delta_{22},$ $\delta_{12}=\pm\nu\delta_{21},\ \delta_{33}=\pm\delta_{44},\ \delta_{43}=\pm\nu\delta_{34}$
$\left\{\begin{array}{l}\alpha_{32}=0,\ \alpha_{21}=\mu\alpha_{43}\\ \alpha_{31}=\mu\alpha_{42},\ \alpha_{41}=\mu\alpha_{42}\alpha_{43}\end{array}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{32}=0,\ \delta_{22}\delta_{33}=\delta_{11}\delta_{44}$ $\mu\delta_{43}\delta_{22}=\delta_{21}\delta_{44},\ \mu\delta_{42}\delta_{33}=\delta_{31}\delta_{44}$; prod. by (23)
$\left\{\begin{array}{l}\alpha_{21}=\mu\alpha_{43},\ \alpha_{32}=\alpha_{43},\\ \alpha_{42}=\frac{1}{2}\alpha_{43}^2,\ \alpha_{31}=\frac{1}{2}\mu\alpha_{43}^2\end{array}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{33}^2=\delta_{22}\delta_{44},\ \delta_{22}^2=\delta_{11}\delta_{33},$ $\delta_{43}\delta_{33}=\delta_{32}\delta_{44},\ \mu\delta_{32}\delta_{22}=\delta_{21}\delta_{33}$
$\left\{\begin{array}{l}\alpha_{21}=\mu\alpha_{43},\ \alpha_{32}=\alpha_{43},\ (p>3)\\ \alpha_{31}=\mu\alpha_{42},\ \alpha_{41}=\mu\alpha_{43}\alpha_{42}-\frac{1}{3}\mu\alpha_{43}^3\end{array}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{22}^2=\delta_{11}\delta_{33},\ \delta_{33}^2=\delta_{22}\delta_{44}.$ $\delta_{33}\delta_{32}=\delta_{43}\delta_{22},\ \delta_{21}\delta_{44}=\mu\delta_{33}\delta_{32},$ $\delta_{42}\delta_{33}\delta_{44}+\delta_{43}\delta_{32}\delta_{44}-\delta_{43}^2\delta_{33}=\mu^{-1}\delta_{31}\delta_{44}^2$
$\left\{\begin{array}{l}\alpha_{32}=0,\ \alpha_{42}=\alpha_{21},\ \alpha_{31}=-\nu_1\alpha_{43}\\ \alpha_{41}=\frac{1}{2}\alpha_{21}^2-\frac{1}{2}\nu_1\alpha_{43}^2\end{array}\right\}$	$\delta_{12}=\delta_{13}=\delta_{14}=\delta_{24}=\delta_{34}=0,$ $\delta_{33}=\pm\delta_{22},\ \delta_{32}=\pm\nu_1\delta_{23}$, and (7)

*Footnote on opposite page.

$B_{41\alpha}$: $\xi_4' = \xi_4 + \alpha\xi_1$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{24} = \delta_{34} = 0$
$B_{21\alpha}B_{43\mu\alpha}$	$\delta_{12} = \delta_{14} = \delta_{32} = \delta_{34} = 0,\ \delta_{11}\delta_{44} = \delta_{22}\delta_{33},$ $\delta_{11}\delta_{42} = \mu\delta_{22}\delta_{31},\ \delta_{22}\delta_{18} = \mu\delta_{11}\delta_{24}$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ \frac{1}{2}\alpha^2 & 0 & \alpha & 1 \end{pmatrix} = S_\alpha = S_1^{\alpha}$	$\delta_{ij} = 0\ (j > i),\ \delta_{32} = 0,$ $\delta_{33}^2 = \delta_{11}\delta_{44},\ \delta_{43}\delta_{33} = \delta_{31}\delta_{44},$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \mu\alpha & 1 & 0 & 0 \\ \frac{1}{2}\mu\alpha^2 & \alpha & 1 & 0 \\ \frac{1}{6}\mu\alpha^3 & \frac{1}{2}\alpha^2 & \alpha & 1 \end{pmatrix} = \Sigma_\alpha = \Sigma_1^{\alpha}$, $p > 3$.	$\delta_{ij} = 0\ (j > i),\ \delta_{22}^2 = \delta_{11}\delta_{33},\ \delta_{33}^2 = \delta_{22}\delta_{44},$ $\delta_{43}\delta_{33} = \delta_{32}\delta_{44},\ \delta_{21}\delta_{33} = \mu\delta_{22}\delta_{32},\ \delta_{31}\delta_{44} = \mu\delta_{42}\delta_{33}$

At the end of the preceding table occur the four types of C_p, with two cases for the second and fourth types. This result is in accord with that given by the theory of canonical forms†; the present form of S_α and Σ_α are more convenient than the types given by the other theory.

Reference is made in the table to the following conditions:

$$(5)\qquad \begin{array}{c} \mu\delta_{12}\delta_{44} + \delta_{11}\delta_{43} = 0,\ \mu\delta_{22}\delta_{34} + \delta_{33}\delta_{21} = 0 \\ \mu^2\delta_{12}\delta_{34} - \mu\delta_{22}\delta_{44} + \mu\delta_{33}\delta_{11} - \delta_{43}\delta_{21} = 0 \end{array}$$

$$(6)\qquad \begin{array}{l} q \neq 0:\ \delta_{32} = -m\delta_{23},\ \delta_{33} = \delta_{22} + q\delta_{23},\ \delta_{22}^2 + q\delta_{22}\delta_{23} + m\delta_{23}^2 = \delta_{11}\delta_{44} \\ q = 0:\ \delta_{32} = \pm m\delta_{23},\ \delta_{33} = \mp\delta_{22},\ \delta_{22}^2 + m\delta_{23}^2 = \delta_{11}\delta_{44} \end{array}$$

$$(7)\qquad \delta_{22}^2 - \nu_1\delta_{23}^2 = \delta_{11}\delta_{44},\ \delta_{21}\delta_{44} = \delta_{42}\delta_{22} - \nu_1\delta_{43}\delta_{23},\ \delta_{31}\delta_{44} = \delta_{42}\delta_{32} - \nu_1\delta_{43}\delta_{33}.$$

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* The conditions besides these four were initially

$$\begin{aligned} \nu\delta_{11}\delta_{44} - \nu^2\delta_{21}\delta_{34} - \nu\delta_{22}\delta_{33} + \delta_{12}\delta_{43} &= 0, \\ \nu\delta_{21}\delta_{44} - \nu\delta_{11}\delta_{34} - \delta_{12}\delta_{33} + \delta_{22}\delta_{43} &= 0, \\ \nu\delta_{21}\delta_{33} - \delta_{11}\delta_{43} - \delta_{12}\delta_{44} + \nu\delta_{22}\delta_{34} &= 0, \\ \delta_{11}\delta_{33} - \delta_{21}\delta_{43} - \delta_{22}\delta_{44} + \delta_{12}\delta_{34} &= 0. \end{aligned}$$

Since δ_{33}, δ_{43}, δ_{34}, δ_{44}, are not all zero, the determinant of the coefficients must vanish, viz.,

$$(\nu\delta_{11}^2 - \delta_{12}^2 - \nu^2\delta_{21}^2 + \nu\delta_{22}^2)^2 - 4\nu^2(\delta_{11}\delta_{22} - \delta_{21}\delta_{12})^2 = 0,$$
$$\{\nu(\delta_{11} - \delta_{22})^2 - (\delta_{12} - \nu\delta_{21})^2\}\{\nu(\delta_{11} + \delta_{22})^2 - (\delta_{12} + \nu\delta_{21})^2\} = 0.$$

Hence $\delta_{11} = \pm\delta_{22}$, $\delta_{12} = \pm\nu\delta_{21}$. The first and second conditions become

$$\delta_{22}(\pm\delta_{44} - \delta_{33}) + \delta_{21}(\pm\delta_{43} - \nu\delta_{34}) = 0,\ \delta_{22}(\pm\delta_{43} - \nu\delta_{34}) + \nu\delta_{21}(\pm\delta_{44} - \delta_{33}) = 0.$$

Hence $\nu(\pm\delta_{44} - \delta_{33})^2 - (\pm\delta_{43} - \nu\delta_{34})^2 = 0$, so that $\delta_{33} = \pm\delta_{44}$, $\delta_{43} = \pm\nu\delta_{34}$.
The third and fourth conditions now reduce to identities.

† Cf. Putnam, *Amer. Journ.* vol. 24 (1902), pp. 319-366. As the context shows, t is a not-fourth power (not merely a not-square). A slight correction is needed on pp. 358-359. If $d = 4$, S is conjugate with S^m if, and only if, m is a square, while 10 (i) and 10″ (i) generate conjugate cyclic groups; likewise, 10′ (i) and 10‴ (i).

5. LEMMA. *Every binary transformation of determinant unity can be generated by the* $S_\lambda = \left(\begin{smallmatrix}1 & 0\\ \lambda & 1\end{smallmatrix}\right)$ *and any* $B = \left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$, $\Delta = \alpha\delta - \beta\gamma \neq 0$, $\gamma \neq 0$.

Indeed, $S_{-\alpha\gamma^{-1}}\, B S_{-\delta\gamma^{-1}} = \begin{pmatrix} 0 & \gamma \\ -\Delta\gamma^{-1} & 0\end{pmatrix}$, which transforms S_λ into $\begin{pmatrix}1 & -\lambda\gamma^2\Delta \\ 0 & 1\end{pmatrix}$.

The subgroups H *of order* p^6N *of* G, $p > 2$.

6. In the notation of §1, we have $h = 6$, $m = 3$, 4, or 5. Let first $m = 5$. Then H has an operator S commutative with G_{p^5} but not with G_{p^6}. By §4, we may take G_{p^5} to be $\{\alpha_{88-1} = 0\}$.

Let G_{p^5} be $\{\alpha_{21} = 0\}$. Then H contains G_{p^6} and an operator (δ_{ij}) with δ_{13}, δ_{14}, δ_{23}, δ_{24}, δ_{34} all zero, $\delta_{12} \neq 0$. Hence (§5), H contains

$$(8) \qquad G_{p^6(p^2-1)}: \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{pmatrix}, \ \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12} = 1.$$

The latter is a subgroup of the following groups:

$$(9) \qquad G_{p^6(p^2-1)^2(p-1)}: \ (\alpha_{ij}),\ \alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = 0,\ \ |\alpha| = 1;$$

$$(10) \qquad G_{p^3(p^3-1)(p^3-p)(p^3-p^2)}: \ (\alpha_{ij}),\ \alpha_{14} = \alpha_{24} = \alpha_{34} = 0,\ \ |\alpha| = 1.$$

We proceed to show that H lies in one of these groups (9), (10), so that its determination follows at once from that of the binary and ternary linear groups. We first prove that a group H, containing (8) and an operator $T = (\beta_{ij})$ in neither (9) nor (10), is identical with the total group G.

(*a*) $\beta_{14} = \beta_{24} = 0$. Then $\beta_{34} \neq 0$, β_{13} and β_{23} are not both zero. Applying on the right of T a transformation on ξ_1, ξ_2, we have $\beta_{13} \neq 0$, $\beta_{34} \neq 0$, $\beta_{14} = \beta_{24} = \beta_{23} = 0$. Applying $\xi_3' = \xi_3 + \lambda\xi_1 + \mu\xi_2$ on the left, we make also $\beta_{11} = \beta_{12} = 0$. Then applying $\xi_4' = \xi_4 + \rho\xi_1 + \sigma\xi_2$ on the left, we make also $\beta_{31} = \beta_{32} = 0$. Applying $\xi_3' = \xi_3 + \alpha\xi_1$, $\xi_4' = \xi_4 + \beta\xi_3 + \gamma\xi_1$ on the right, we make also $\beta_{33} = 0$, $\beta_{43} = \beta_{44} = 0$. Applying on the left a binary on ξ_1, ξ_2, we make $\beta_{22} = \beta_{41} = 0$, and reach

$$T_1:\ \xi_1' = \beta_{13}\xi_3,\ \xi_2' = \beta_{21}\xi_1,\ \xi_3' = \beta_{34}\xi_4,\ \xi_4' = \beta_{42}\xi_2.$$

Now (8) contains every B_{12}, B_{21}, B_{31}, B_{32}, B_{41}, B_{42}, B_{43}. But T_1 transforms B_{42} into B_{34}, B_{12} into B_{24}, B_{32} into B_{14}, B_{34} into B_{13}, B_{14} into B_{23}. Hence H contains every B_{ij}, so that $H = G$.

(*b*) β_{14}, β_{24} not both zero. Applying $\xi_1' = \xi_2$, $\xi_2' = -\xi_1$ on the right, we have $\beta_{14} \neq 0$. Applying $\xi_i' = \xi_i + \lambda_i\xi_1$ $(i = 2, 3, 4)$ on the right, we

make $\beta_{i4}=0$. Applying $\xi_4'=\xi_4+\alpha_1\xi_1+\alpha_2\xi_2+\alpha_3\xi_3$ on the left, we make $\beta_{11}=\beta_{12}=\beta_{13}=0$.

(b_1) Let $\beta_{23}\neq 0$. Applying $\xi_i'=\xi_i+\lambda_i\xi_2$ $(i=3, 4)$ on the right, and $\xi_3'=\xi_3+\gamma_1\xi_1+\gamma_2\xi_2$ on the left, we make $\beta_{21}=\beta_{22}=\beta_{33}=\beta_{43}=0$. Applying on the left a binary on ξ_1, ξ_2, we make also $\beta_{31}=\beta_{42}=0$. There results $(\xi_1\xi_4)(\xi_2\xi_3)\cdot M$, M multiplying each ξ_i by a constant. This transforms B_{21}, B_{31}, B_{41}, B_{32}, B_{42} into B_{34}, B_{24}, B_{14}, B_{23}, B_{13}, respectively. Hence $H=G$.

(b_2) Let $\beta_{23}=0$. Applying on the left a binary on ξ_1, ξ_2, we make $\beta_{22}=0$. Applying on the right $B_{i2}(i=3, 4)$, we make $\beta_{i1}=0$. If $\beta_{33}\neq 0$, we apply B_{43} on the right and B_{32} on the left and make $\beta_{43}=\beta_{32}=0$, and reach $(\xi_1\xi_4\xi_2)\cdot M$. If $\beta_{33}=0$, we apply B_{32} on the left and make $\beta_{42}=0$, reaching $(\xi_1\xi_4\xi_3\xi_2)\cdot M$. In either case, $H=G$.

Next, let H contain the subgroup (8), and an operator (β) in (9) but not in (10), and an operator (β') in (10), but not in (9). From (β) and (8) we readily obtain (§5) all binary operators on ξ_3, ξ_4. But the product of (β') by $\xi_3'=\xi_4$, $\xi_4'=-\xi_3$ gives an operator in neither (9) nor (10). Hence $H=G$.

7. Let next G_{p^6} be $\{\alpha_{43}=0\}$. Then H contains

$$(11)\qquad G'_{p^6(p^2-1)}:\begin{pmatrix}1 & 0 & 0 & 0\\ \alpha_{21} & 1 & 0 & 0\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34}\\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44}\end{pmatrix},\ \alpha_{33}\alpha_{44}-\alpha_{43}\alpha_{34}=1\,.$$

The latter lies in (9) and

$$(12)\qquad G'_{p^3(p^3-1)(p^3-p)(p^3-p^2)}:\ (\alpha_{ij}),\ \alpha_{12}=\alpha_{13}=\alpha_{14}=0\,,\ |\alpha|=1\,.$$

We show that H lies in (9) or (12). This follows from the result of §6 and the fact that the matrices of the general operators of (8), (9), (10) become the matrices of the general operators of (11), (9), (12), respectively, upon reflection on their left diagonals.

Indeed, every group of n-ary transformations*

$$(\alpha):\qquad \xi_i'=\sum_{j=1}^{n}\alpha_{ij}\xi_j\qquad (i=1,\ldots,n)$$

is simply isomorphic with the group of the transformations $\{\alpha\}^{-1}$, inverse of

$$\{\alpha\}:\qquad \xi_i'=\sum_{j=1}^{n}\alpha_{n+1-j\ n+1-i}\,\xi_j\qquad (i=1,\ldots,n),$$

* We thus obtain without computation the result of Burnside on the ternary groups (a_{ij}), $a_{12}=a_{13}=0$, and (a_{ij}), $a_{13}=a_{23}=0$, *Proc. L. M. S.*, vol. 26 (1894), pp. 94-95.

where the matrix of $\{\alpha\}$ is obtained by reflecting the matrix of (α) on its left diagonal. We have only to show that

$$\text{if } (\alpha) \sim \{\alpha\}^{-1},\ (\beta) \sim \{\beta\}^{-1}, \text{ then } (\alpha)(\beta) \sim \{\alpha\}^{-1}\{\beta\}^{-1} = (\{\beta\}\{\alpha\})^{-1}.$$

This is true since

$$(\alpha)(\beta): \qquad \xi_i' = \sum_{k=1}^{n} \gamma_{ik}\xi_k, \ \gamma_{ik} \equiv \sum_{j=1}^{n} \beta_{ij}\alpha_{jk} \qquad (i, k = 1, \ldots, n),$$

$$\{\beta\}\{\alpha\}: \qquad \xi_i' = \sum_{k=1}^{n} \gamma_{ik}'\xi_k, \ \gamma_{ik}' \equiv \sum_{j=1}^{n} \beta_{n+1-k\, n+1-j}\alpha_{n+1-j\, n+1-i},$$

$$\gamma_{ik}' = \sum_{j_1=1}^{n} \beta_{n+1-k j_1}\alpha_{j_1\, n+1-i} = \gamma_{n+1-k\, n+1-i}.$$

8. Let finally G_{p^5} be $\{\alpha_{32} = 0\}$. Then H contains

$$(13) \qquad G''_{p^6(p^2-1)}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{pmatrix}, \ \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} = 1.$$

This lies in (10) and (12). By a proof of character similar to that in §6, making the separation of cases $\beta_{14} = 0$, $\beta_{14} \neq 0$, it is found that, if H contains (13) either $H = G$, or else H lies in (10) or (12).

9. Let next $m = 4$ $(h = 6)$. By §4, we may take $G_{p^4} = \{\alpha_{21} = \alpha_{43} = 0\}$. Then H contains an operator S with $\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0$, δ_{12}, δ_{34} not both zero. Now S transforms $[\alpha_{ij}]$ into (β_{ij}), with $\beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = 0$,

$$\beta_{11} = 1 + \alpha_{21}\delta_{12}\delta_{22}\Delta_1^{-1}, \ \beta_{12} = -\alpha_{21}\delta_{12}^2\Delta_1^{-1}, \ \beta_{21} = \alpha_{21}\delta_{22}^2\Delta_1^{-1},$$
$$\beta_{22} = 1 - \alpha_{21}\delta_{12}\delta_{22}\Delta_1^{-1}, \qquad \Delta_1 \equiv \delta_{11}\delta_{22} - \delta_{12}\delta_{21},$$

the expressions for $\beta_{i+2\,j+2}$ being derived from those for β_{ij} $(i \overline{\gtrless} 2,\ j \overline{\gtrless} 2)$ by adding 2 to each subscript. Hence if $\delta_{12} = 0$, G_{p^6} and $S^{-1}G_{p^6}S$ have $\{\alpha_{43} = 0\}$ in common, contrary to $m = 4$. Hence $\delta_{12} \neq 0$, and similarly, $\delta_{34} \neq 0$. Then, by §5, H contains the group (9). But (9) is a maximal subgroup of G (§6).

10. Let finally $m = 3$ $(h = 6)$. By §1, the order of H must be a multiple of $p^6(p^4 - 1)(p^3 - 1)$. But* G has no subgroup of index $< p^3 + p^2 + p + 1$ or 27, according as $p > 3$, or $p = 3$. Similarly, the case $m = 2$, $h = 5$ is excluded, since $p(p^2 - 1) < p^3$. A partial summary of our results is given by the

* *Transactions A. M. S.*, vol. 6 (1905), pp. 48-57.

THEOREM. *The only maximal subgroups of order a multiple of* p^6 *are conjugate with* (9) *or with the simply isomorphic groups* (10) *and* (12).

The subgroups H of order p^5N, *N prime to* p, $p > 2$.

11. Let $m = 4$ $(h = 5)$. We may set $G_{p^4} = \{\alpha_{21} = \alpha_{43} = 0\}$; otherwise, there is (§1) an operator S commutative with G_{p^4}; it has every $\delta_{ij} = 0$ $(j > i)$ and hence transforms G_{p^5} into a subgroup G'_{p^5} of G_{p^6}, where G'_{p^5} and G_{p^5} are distinct and hence generate G_{p^6}, contrary to $h = 5$. Thus H contains an operator $S = (\delta_{ij})$, with $\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0$, which is commutative with $G_{p^4} = \{\alpha_{21} = \alpha_{43} = 0\}$ and not with G_{p^5}. The latter may be taken to be $\{\alpha_{21} = 0\}$, $\{\alpha_{43} = 0\}$ or $\{\alpha_{21} = \mu\alpha_{43}\}$. In view of §7, we treat only one of the first two cases.

(*a*) Let G_{p^5} be $\{\alpha_{43} = 0\}$. Then S transforms $[\alpha]$, with $\alpha_{43} = 0$, into $B = (\beta_{ij})$, where β_{11}, β_{12}, β_{21}, β_{22} are given in §9, while β_{13}, β_{14}, β_{23}, β_{24}, β_{43}, β_{34} are all zero, $\beta_{33} = \beta_{44} = 1$. Hence $\delta_{12} \neq 0$. Multiplying B on the left by a suitably chosen operator of G_{p^4}, the product becomes a binary transformation $\begin{pmatrix}\beta_{11}\beta_{12}\\ \beta_{21}\beta_{22}\end{pmatrix}$ on ξ_1, ξ_2, with $\beta_{12} \neq 0$. This with the $\begin{pmatrix}1 & 0\\ \alpha_{21} & 1\end{pmatrix}$ generate every binary transformation on ξ_1, ξ_2, of determinant 1 (§5). Hence H contains

(14) $\qquad G_{p^5(p^2-1)}$: operators (8) with $\beta_{43} = 0$.

Suppose that H contains an operator (β_{ij}) not in (14).

(a_1) If in every such (β_{ij}), $\beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = 0$, we apply on the right a binary on ξ_1, ξ_2 and make $\beta_{11} = 1$, $\beta_{12} = \beta_{21} = 0$, then apply B_{41}, B_{42}, B_{31}, B_{32} and reach DC, where C is $\xi'_2 = \beta\xi_2$, $\xi'_3 = \beta^{-1}\xi_3$, and D is a binary on ξ_3, ξ_4 of determinant 1. The operators of H are all of the form $DC\Gamma$, Γ in (14). The operators D must form a group of order prime to p (since $h = 5$) and hence of order c, $4d$, 24, 48, 120, being cyclic, dicyclic, or the homogeneous forms of the groups of the regular bodies. The form of H is thus evident.

(a_2) If in one $B = (\beta_{ij})$, β_{13}, β_{14}, β_{23}, β_{24} are not all zero, the order of H will be shown to be a multiple of p^6, contrary to $h = 5$. We may take β_{13}, β_{14} not both zero, using BB_{12} if necessary. If $\beta_{13} = 0$, we normalize by transforming by $\xi'_3 = \xi_4$, $\xi'_4 = -\xi_3$, so that (14) is unaltered. Hence we have $\beta_{14} \neq 0$. Applying B_{i1} on the right, we make $\beta_{i4} = 0$ $(i = 2, 3, 4)$. Applying B_{41}, B_{42} on the left, we make also $\beta_{11} = \beta_{12} = 0$. If $\beta_{13} \neq 0$, we normalize by transformation by B_{43}, so that (14) is unaltered, and make $\beta_{13} = 0$. If $\beta_{23} \neq 0$, we may make $\beta_{21} = \beta_{22} = \beta_{33} = \beta_{43} = 0$. Applying on the left a binary on ξ_1, ξ_2, we reach

$(\xi_1\xi_4)(\xi_2\xi_3)\cdot M$. This transforms B_{12} into B_{43}, which extends $\{\alpha_{43}\}$ to G_{p^6}. If $\beta_{23}=0$, we may make $\beta_{22}=\beta_{31}=\beta_{41}=0$. If then $\beta_{33}\neq 0$, we may make $\beta_{32}=0$, $\beta_{42}=1$, and reach

$$\xi_1'=\beta_{14}\xi_4,\ \xi_2'=\beta_{21}\xi_1,\ \xi_3'=\beta_{33}\xi_3,\ \xi_4'=\xi_2+\beta_{43}\xi_3.$$

This transforms $\xi_3'=\xi_3+\xi_2$ into

$$\xi_1'=\xi_1,\ \xi_2'=\xi_2,\ \xi_3'=(1-\beta_{43})\,\xi_3+\beta_{33}\xi_4,\ \xi_4'=-\frac{\beta_{43}^2}{\beta_{33}}\xi_3+(1+\beta_{43})\,\xi_4.$$

This is commutative with $\{\alpha_{43}=0\}$, extenting it to G_{p^6}. Finally, if $\beta_{33}=0$, we make $\beta_{42}=0$ and reach $(\xi_1\xi_4\xi_3\xi_2)\cdot M$. Its square transforms B_{21} into B_{43}.

(b) Let G_{p^5} be $\{\alpha_{21}=\mu\alpha_{43}\}$. Then δ_{12}, δ_{34} are not both zero in S.

(b_1) Let first $\delta_{34}=0$, $\delta_{12}\neq 0$. We transform $[\alpha]$, with $\alpha_{21}=\mu\alpha_{43}\neq 0$, by S and apply on the left a suitably chosen operator of G_{p^5} and obtain a binary $B=\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}$ on ξ_1, ξ_2 of determinant 1 and having $\gamma\neq 0$. Now G_{p^5} contains $B_{21\lambda}B_{43\lambda\mu^{-1}}\equiv\Sigma_\lambda$. As in §5,

$$\Sigma_{-\alpha\gamma^{-1}}\,B\Sigma_{-\delta\gamma^{-1}}=\begin{pmatrix}0 & \gamma\\ -\gamma^{-1} & 0\end{pmatrix}_{1,2}B_{43\rho},\ \rho\equiv-(\alpha+\delta)\gamma^{-1}\mu^{-1}.$$

This transforms Σ_λ into $\begin{pmatrix}1 & -\lambda\gamma^2\\ 0 & 1\end{pmatrix}_{1,2}B_{4,3,\lambda\mu^{-1}}\equiv U_{\lambda,\gamma}$. Then

$$U_{\lambda,\gamma}\Sigma_\lambda^{-1}=\begin{pmatrix}1 & -\lambda\gamma^2\\ -\lambda & 1+\lambda^2\gamma^2\end{pmatrix}_{1,2}.$$

Hence H contains a binary transformation B in which γ is arbitrary, and hence also $U_{1,\frac{1}{2}(r-1)}\,U^{-1}_{1,\frac{1}{2}(r+1)}\equiv B_{1,2,r}$, r arbitrary. Then from $U_{\lambda,\gamma}$, we obtain every B_{43} in H, and hence also B_{21}. Hence H contains G_{p^6}.

(b_2) The case $\delta_{12}=0$, $\delta_{34}\neq 0$ is excluded similarly, or by §7.

(b_3) Let finally $\delta_{12}\neq 0$, $\delta_{34}\neq 0$. As in (b_1), we readily obtain from S a transformation B_1B_2, where $B_1=\begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{pmatrix}_{1,2}$, of determinant 1, having $\beta_{12}\neq 0$, and $B_2=\begin{pmatrix}\beta_{33} & \beta_{34}\\ \beta_{43} & \beta_{44}\end{pmatrix}_{3,4}$, of determinant 1, having $\beta_{34}\neq 0$. We proceed on the latter as in (b_1) and reach $C=C_1C_2$, where $C_1=\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}$, $\gamma\neq 0$, $C_2=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. We transform by $\xi_2'=\xi_2+\rho\xi_1$, which transforms G_{p^4} and G_{p^5} into themselves, and make $\alpha=\gamma\mu$. Then

$$T\equiv\Sigma_{-\mu}C\Sigma_{-\delta\gamma^{-1}}=\begin{pmatrix}0 & \gamma\\ -\gamma^{-1} & 0\end{pmatrix}_{1,2}\begin{pmatrix}0 & 1\\ -1 & b\end{pmatrix}_{3,4},\ b=1-\frac{\delta}{\mu\gamma}.$$

Then $T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_{1,2} \begin{pmatrix} -1 & b \\ -b & b^2-1 \end{pmatrix}_{3,4}$. Hence $b=0$ by (b_2). Then T transforms $\Sigma_{-\mu}$ into $B_{1,2,\mu\gamma^2} B_{3,4,1} \equiv W$. Then $W^{-1} B_{2,1,\mu} B_{4,3,1} WT$ is of the form $B_1 B_2$, with $\beta_{12} = \gamma\,(1-\mu^2\gamma^2)$, $\beta_{34}=0$. Hence $\mu^2\gamma^2 = 1$ by (b_1). The group generated by Σ_λ and W is composed of the operators

$$(15)\qquad \begin{pmatrix} \alpha & \gamma\mu^{-1} \\ \beta\mu & \delta \end{pmatrix}_{1,2} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}_{3,4}, \quad \alpha\delta - \beta\gamma = 1.$$

These with $\{\alpha_{21} = \alpha_{43} = 0\}$ generate a $G'_{p^5(p^2-1)}$.

Let H have a further operator $B = (\beta_{ij})$. If $\beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = 0$, we reach $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{1,2} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}_{3,4}$. Then $b=0$ by (b_1). Hence H contains $Y = \begin{pmatrix} a & 0 \\ c & a^{-1}e^{-1} \end{pmatrix}_{1,2}$ $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}_{3,4}$. Let first $c \neq 0$. Then $B_{12,\lambda\mu^{-1}} B_{3,4,\lambda}$ transforms Y into $\begin{pmatrix} \cdot & r \\ c & \cdot \end{pmatrix}_{1,2}$ $\begin{pmatrix} e & \lambda - \lambda e \\ 0 & 1 \end{pmatrix}_{3,4}$, $r = \frac{\lambda}{\mu}\,(a^{-1}e^{-1} - a - \frac{\lambda}{\mu} c)$. If $e \neq a^{-2}$, we can take $\lambda \neq 0$ to make $r=0$. Then $e=1$ by (b_2), and

$$\Sigma_{-\lambda\mu} Y^{-1} \Sigma_{\lambda\mu} Y = \begin{pmatrix} 1 & 0 \\ \lambda\mu\,(a^{-2}-1) & 1 \end{pmatrix}_{1,2},$$

whence $a^{-2} = 1 = e$ by (b_1). There remains the case $e = a^{-2}$. Then

$$Y^{-1} \Sigma_{\lambda\mu}^{-1} Y \Sigma_{\lambda\mu} = \begin{pmatrix} 1 & 0 \\ \lambda\,(1-a^2) & 1 \end{pmatrix}_{3,4},$$

whence $a^2 = 1$. Then $Y^2 = B_{2,1,\pm 2c}$, giving a G_{p^6}. Hence must $c=0$ in Y. Then

$$\Sigma_{-e^{-1}\mu} Y^{-1} \Sigma_\mu Y = B_{2,1,\rho}, \quad \rho = \mu e^{-1}(a^{-2}-1),$$

whence $a^2 = 1$. Then $Y = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm e^{-1} \end{pmatrix}_{1,2} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}_{3,4}$ transforms $G'_{p^5(p^2-1)}$ into itself and extends it to a group of order $p^5(p^2-1)t$, t a factor of $2(p-1)$. Replacing α, γ by $e^{-1}\alpha$, $e^{-1}\gamma$, in the product of (15) by Y, we obtain (19).

Suppose next that β_{13}, β_{14}, β_{23}, β_{24} are not all zero in B. We readily make $\beta_{14} = 1$, β_{11}, β_{12}, β_{13}, β_{24}, β_{34}, β_{44} all zero,

Let $\beta_{23} \neq 0$. We readily make $\beta_{21} = \beta_{22} = \beta_{33} = \beta_{43} = 0$. If $\beta_{32} = 0$, $\beta_{41} = 0$, then $B = (\xi_1\xi_4\xi_2\xi_3)\,M$ and $H = G$. If $\beta_{32} = 0$, $\beta_{41} \neq 0$, then $BT_\lambda BT_\rho$, where $T_\lambda \equiv B_{1,2,\lambda\mu^{-1}} B_{3,4,\lambda}$, has the form

$$(16)\qquad \begin{pmatrix} g & h \\ j & k \end{pmatrix}_{1,2} \begin{pmatrix} u & v \\ w & z \end{pmatrix}_{3,4},$$

$h = \beta_{42}(1 + \mu^{-1}\rho\lambda\beta_{23})$, $v = \beta_{31} + \rho\beta_{41}$. We may choose ρ and λ to make $v = 0$, $h \neq 0$, contrary to (b_1). Hence $\beta_{32} \neq 0$. Then $(B\Sigma_{\lambda\mu})^2$ is of the form (16) with $h = \beta_{42} + \lambda\beta_{32}$, $v = \beta_{31} + \lambda\mu\beta_{32}$; hence $\beta_{31} = \mu\beta_{42}$ by (b_2). If $\beta_{42} = 0$, $B = (\xi_1\xi_4)(\xi_2\xi_3)\cdot M$, and $H = G$. Hence B has the form

$$(17) \qquad \xi_1' = \xi_4,\ \xi_2' = c\xi_3,\ \xi_3' = \mu d\xi_1 + b\xi_2,\ \xi_4' = e\xi_1 + d\xi_2,\ cbd \neq 0.$$

Then $\{(15)\,B\}^2$ is of the form (16) with

$$h = \beta\,(d\gamma + b\delta) + \mu^{-1}e\gamma\delta + d\delta^2,\ v = \mu d\alpha\delta + \mu b\beta\delta + bc\gamma\delta + cd\gamma^2.$$

If $\delta \neq 0$, we can choose α to satisfy $\alpha\delta - \beta\gamma = 1$ and $v = 0$ if

$$(18) \qquad \mu d + (\mu\beta + c\gamma)(d\gamma + b\delta) = 0\,.$$

We can give to $d\gamma + b\delta$ any value $t \neq 0$ by choice of γ and then satisfy (18) by choice of β. The resulting value of h is

$$-d + \frac{\delta t}{d\mu}(e + bc) - \frac{ct^2}{d\mu} + \delta^2\left(d - \frac{be}{d\mu}\right).$$

By (b_1) this must be zero for every $t \neq 0$, $\delta \neq 0$. Hence $p = 3$, $e = -bc$. The determinant of (17) is $-c\,(\mu d^2 - be) = 1$. Hence $p = 3$, $c = \mu$, $e = -\mu b$. Then the square of (17) is

$$L \equiv \begin{pmatrix} -\mu b & d \\ d & \mu b \end{pmatrix}_{1,2} \begin{pmatrix} \mu b & \mu d \\ \mu d & -\mu b \end{pmatrix}_{3,4}$$

The product of (15) with $\alpha = \mu b$, $\gamma = \mu d$, $\beta = \mu d$, $\delta = -\mu b$, by L^{-1} is $V^{\pm 1}$,

$$V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{1,2};\ \Sigma_1^{-1}V^{-1}\Sigma_1 V^{-1} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}_{1,2},$$

contrary to (b_1).

12. Let $m = 3$ $(h = 5)$. The 5 possible G_{p^3} are studied under (a) — (e).

(a) Let G be $\{\alpha_{21} = \alpha_{32} = \alpha_{31} = 0\}$. Then H has an operator $S = (\delta_{ij})$ with $\delta_{14} = \delta_{24} = \delta_{34} = 0$, δ_{12}, δ_{13}, δ_{23} not all zero. As the G_{p^5} we may take $\{\alpha_{21} = 0\}$, $\{\alpha_{32} = 0\}$, or $\{\alpha_{21} = \alpha_{32}\}$.

(a_1) Let G_{p^5} be $\{\alpha_{21} = 0\}$. Then δ_{13}, δ_{23} are not both zero; otherwise S transforms G_{p^5} into itself (§9, or §4). We may normalize by transforming by $\xi_1' = \xi_2$, $\xi_2' = -\xi_1$, and set $\delta_{13} \neq 0$. Applying operators of G_{p^6}, we may make δ_{11}, δ_{12}, δ_{33}, δ_{41}, δ_{42}, δ_{43} all zero. Normalizing by transformation by B_{21}, we make also $\delta_{23} = 0$. If $\delta_{22} \neq 0$, SB_{32} has $\delta_{32} = 0$, and transforms B_{31} into B_{13}. The commutator of B_{13}, B_{32} is B_{12}. But G_{p^5} and B_{12} generate a G'_{p^6}, the transform

of G_{p^6} by $(\xi_1\xi_2)$. If $\delta_{22}=0$, SB_{32} has $\delta_{31}=0$, giving $(\xi_1\xi_3\xi_2)\cdot M$. Its square transforms B_{32} into B_{21}, giving G_{p^6}.

(a_2) Let G_{p^5} be $\{\alpha_{32}=0\}$. Then δ_{12}, δ_{13} are not both zero (§4). Normalizing by transformation by $\xi_2'=\xi_3$, $\xi_3'=-\xi_2$, we may set $\delta_{13}\neq 0$. We readily make $\delta_{11}=\delta_{23}=\delta_{33}=\delta_{41}=\delta_{42}=\delta_{43}=0$. Normalizing by B_{32}, we make $\delta \;=0$. If $\delta_{22}=0$, $B_{21}S$ has $\delta_{31}=0$, giving $(\xi_1\xi_3\xi_2)\cdot M$. This transforms B_{21} into B_{32}, giving G_{p^6}. If $\delta_{22}\neq 0$, $B_{21}S$ has $\delta_{21}=0$, and transforms $B_{2,1,1}$ into

$$\xi_2'=(1-\delta_{32}\delta_{31}^{-1})\,\xi_2+\delta_{22}\delta_{31}^{-1}\xi_3,\; \xi_3'=-\delta_{32}^2\delta_{31}^{-1}\delta_{22}^{-1}\xi_2+(1+\delta_{32}\delta_{31}^{-1})\,\xi_3,$$

which is commutative with $\{\alpha_{32}=0\}$ by §4 and extends it to a G'_{p^6}.

(a_3) Let G_{p^5} be $\{\alpha_{21}=\alpha_{32}\}$. The operators in H with $\delta_{14}=\delta_{24}=\delta_{34}=0$ form a subgroup H_1 containing G_{p^5}. The partial transformations on ξ_1, ξ_2, ξ_3 form a group H' of order p^2l, l prime to p. It contains G_{p^2}: $\xi_2'=\xi_2+\alpha\xi_1$, $\xi_3'=\xi_3+\alpha\xi_2+\beta\xi_1$. The group H_1' of the operators of determinant 1 of H' contain G_{p^2} self-conjugately.* This is also true of H' since the number of conjugates to G_{p^2} in H' is a divisor of $p-1$. Hence $\delta_{12}=\delta_{13}=\delta_{23}=0$ in every operator (δ_{ij}) of H', contrary to (a).

(b) Let G_{p^3} be $\{\alpha_{32}=\alpha_{43}=\alpha_{42}=0\}$. This case is reduced to case (a) by means of the general method of §7.

(c) Let G_{p^3} be $\{\alpha_{21}=\alpha_{43}=0,\alpha_{41}=\mu\alpha_{32}\}$. Then H has an operator $S=(\delta_{ij})$, with $\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=0$. Now G_{p^5} may be taken to be $\{\alpha_{21}=0\}$, $\{\alpha_{43}=0\}$, or $\{\alpha_{21}=\mu\alpha_{43}\}$. But in each case, G_{p^5} and $S^{-1}G_{p^5}S$ have $\{\alpha_{21}=\alpha_{43}=0\}$ in common (§9), contrary to $m=3$.

(d) Let G_{p^3} be $\{\alpha_{32}=0,\alpha_{21}=\mu\alpha_{43},\alpha_{31}=-\mu\alpha_{42}\}$. Then H has an operator $S=(\delta_{ij})$, with $\delta_{12}=\delta_{13}=\delta_{14}=\delta_{24}=\delta_{34}=0$, $\delta_{23}\neq 0$, $\delta_{22}\delta_{33}-\delta_{23}\delta_{32}=\delta_{11}\delta_{44}=\pm 1$. Since $\{\alpha_{32}=0\}$ is excluded (§4), G_{p^5} may be taken to be $\{\alpha_{21}=\mu\alpha_{43}\}$ or $\{\alpha_{21}=\alpha_{32}+\mu\alpha_{43}\}$. In either case we apply on the right and left of S operators of G_{p^5} and get

$$D\colon\; \xi_1'=\delta_{11}\xi_1,\; \xi_2'=\delta_{23}\xi_3,\; \xi_3'=\delta_{31}\xi_1\mp\delta_{23}^{-1}\xi_2,\; \xi_4'=\pm\delta_{11}^{-1}\xi_4.$$

This transforms B_{421} into $\xi_4'=\xi_4-\delta_{23}\delta_{11}^{-1}\xi_3+\delta_{23}\delta_{31}\delta_{11}^{-2}\xi_1$. Hence H contains B_{43} and hence also G_{p^6}.

(e) Let G_{p^3} be $\{\alpha_{32}=0,\alpha_{42}=\alpha_{21},\alpha_{31}=q\alpha_{21}+m\alpha_{43}\}$. Then G_{p^5} is $\{\alpha_{32}=0\}$. But every operator commutative with G_{p^3} is commutative with G_{p^5} (§4).

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* *American Journ. Math.*, vol. 26 (1905), p. 193, §7.

THEOREM. *A subgroup of order p^5N, N prime to p, $p > 2$, either has an invariant G_{p^5}, when its form is readily determined from* §4, *or is conjugate with one of the following types of groups*: (9) *with the* $\begin{pmatrix} \alpha_{11}\alpha_{12} \\ \alpha_{21}\alpha_{22} \end{pmatrix}$ *defining a binary group of order prime to p*; (9) *with the same restriction on the* $\begin{pmatrix} \alpha_{33}\alpha_{34} \\ \alpha_{43}\alpha_{44} \end{pmatrix}$; *a group of $p^5(p^2-1)t$ operators, t a factor of* $2\ (p-1)$:

$$(19) \qquad \begin{pmatrix} \pm\alpha e^{-1} & \pm\mu^{-1}\gamma e^{-1} & 0 & 0 \\ \pm\mu\beta e^{-1} & \pm\delta e^{-1} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & \alpha & \gamma \\ \alpha_{41} & \alpha_{42} & \beta & \delta \end{pmatrix}, \quad \begin{array}{l} \alpha\delta - \beta\gamma = e, \\ \mu \textit{ as in } \S 4. \end{array}$$

13. The results in the earlier sections suffice for a similar treatment of the subgroups of order p^4N; also for those of order p^3N, special considerations being necessary for the case in which no two G_{p^3} have a common operator other than identity, N being a multiple of $(p^4-1)(p^3-1)$.

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